Integration 1

1. Evaluate $\int_{-1}^{1} \frac{1+x^{2016}}{1+2^x} dx$.

Let $I = \int_{-1}^{1} \frac{1+x^{2016}}{1+2^x} dx \quad ....(1)$

Put $y = -x$, then

$$I = \int_{1}^{-1} \frac{1+(-y)^{2016}}{1+2^{-y}} d(-y) = -\int_{1}^{-1} \frac{1+y^{2016}}{1+2^{-y}} dy = \int_{-1}^{1} \frac{1+y^{2016}}{1+2^{-y}} dy = \int_{-1}^{1} \frac{1+x^{2016}}{1+2^{-x}} dx \quad ....(2)$$

(1) + (2), $\quad 2I = \int_{-1}^{1} \frac{1+x^{2016}}{1+2^x} dx + \int_{-1}^{1} \frac{1+x^{2016}}{1+2^{-x}} dx = \int_{-1}^{1} \frac{1+x^{2016}}{1+2^x} dx + \int_{-1}^{1} \frac{(1+x^{2016})2^x}{1+2^x} dx$

$$= \int_{-1}^{1} \frac{1+x^{2016}}{1+2^x} + \frac{(1+x^{2016})2^x}{1+2^x} dx = \int_{-1}^{1} \frac{1+x^{2016}(1+2^x)}{1+2^x} dx = \int_{-1}^{1} (1+x^{2016}) dx$$

$$\int_{-1}^{1} (1+x^{2016}) dx = x + \frac{x^{2017}}{2017}\Big]_{-1}^{1} = \frac{4036}{2017}$$

$\therefore\ I = \mathbf{\frac{2018}{2017}}$

2. **(a)** Consider the integrals: $\quad I_1 = \int \frac{\cos x}{\cos x + \sin x} dx$ and $I_2 = \int \frac{\sin x}{\cos x + \sin x} dx$

By evaluating $\ I_1 + I_2\ $ and $\ I_1 - I_2\ $, find the values of $\ I_1\ $ and $\ I_2\ $.

(b) Evaluate $\quad \int \frac{\cos x}{a\cos x + b\sin x} dx$, where $\ a^2 + b^2 \neq 0\ $.

(c) Evaluate $\quad \int \frac{e^x}{a e^x + b e^{-x}} dx$, where $\ a, b \neq 0\ $.

(a) $I_1 + I_2 = \int \frac{\cos x}{\cos x + \sin x} dx + \int \frac{\sin x}{\cos x + \sin x} dx = \int \frac{\cos x + \sin x}{\cos x + \sin x} dx = \int dx = x + k_1$

$I_1 - I_2 = \int \frac{\cos x - \sin x}{\cos x + \sin x} dx = \int \frac{d(\cos x + \sin x)}{\cos x + \sin x} = \ln|\cos x + \sin x| + k_2$

Solving the above simultaneous equations, we get:

$$I_1 = \frac{1}{2}(x + \ln|\cos x + \sin x|) + c_1$$

$$I_2 = \frac{1}{2}(x - \ln|\cos x + \sin x|) + c_2$$

(b) Let $\quad I_3 = \int \frac{\cos x}{a\cos x + b\sin x} dx$ and $I_4 = \int \frac{\sin x}{a\cos x + b\sin x} dx$

Then $\quad aI_3 + bI_4 = \int \frac{a\cos x + b\sin x}{a\cos x + b\sin x} dx = \int dx = x + k_3$

$$bI_3 - aI_4 = \int \frac{b\cos x - a\sin x}{a\cos x + b\sin x}dx = \int \frac{d(a\cos x + b\sin x)}{a\cos x + b\sin x} = \ln|a\cos x + b\sin x| + k_4$$

Solving the above simultaneous equations, we get:

$$I_3 = \frac{1}{a^2+b^2}(ax + b\ln|a\cos x + b\sin x|) + c_3$$

$$I_4 = \frac{1}{a^2+b^2}(ax - b\ln|a\cos x + b\sin x|) + c_4$$

(c) Let $I_5 = \int \frac{e^x}{a\,e^x + b\,e^{-x}}dx$ and $I_6 = \int \frac{e^{-x}}{a\,e^x + b\,e^{-x}}dx$

Then $aI_5 + bI_6 = \int \frac{a\,e^x + b\,e^{-x}}{a\,e^x + b\,e^{-x}}dx = \int dx = x + k_5$

$$aI_5 - bI_6 = \int \frac{a\,e^x - b\,e^{-x}}{a\,e^x + b\,e^{-x}}dx = \int \frac{d(a\,e^x + b\,e^{-x})}{a\,e^x + b\,e^{-x}} = \ln|a\,e^x + b\,e^{-x}| + k_6$$

Solving the above simultaneous equations, we get:

$$I_5 = \frac{1}{2a}(x + \ln|a\,e^x + b\,e^{-x}|) + c_5$$

$$I_6 = \frac{1}{2b}(x - \ln|a\,e^x + b\,e^{-x}|) + c_6$$

3. **(a)** By using the substitution $x = \alpha\cos^2\theta + \beta\sin^2\theta$, evaluate the integral

$\int_\alpha^\beta \frac{1}{\sqrt{(x-\alpha)(\beta-x)}}dx$, where $\alpha < \beta$.

Evaluate this integral where $\alpha > \beta$.

(b) By using the substitution $t = \frac{1}{x}$, evaluate the integral

$\int_a^b \frac{1}{t\sqrt{(t-a)(b-t)}}dt$, where $0 < a < b$.

(a) $I = \int_\alpha^\beta \frac{1}{\sqrt{(x-\alpha)(\beta-x)}}dx$ and $x = \alpha\cos^2\theta + \beta\sin^2\theta$

$$\begin{aligned}(x-\alpha)(\beta-x) &= (\alpha\cos^2\theta + \beta\sin^2\theta - \alpha)(\beta - \alpha\cos^2\theta - \beta\sin^2\theta)\\ &= [-\alpha(1-\cos^2\theta) + \beta\sin^2\theta][\beta(1-\sin^2\theta) - \alpha\cos^2\theta]\\ &= [-\alpha\sin^2\theta + \beta\sin^2\theta][\beta\cos^2\theta - \alpha\cos^2\theta]\\ &= (\beta-\alpha)^2\sin^2\theta\cos^2\theta\end{aligned}$$

$$\frac{dx}{d\theta} = -2\alpha\cos\theta\sin\theta + 2\beta\cos\theta\sin\theta = 2(\beta-\alpha)\cos\theta\sin\theta$$

(i) When $x = \alpha$, $\alpha = \alpha\cos^2\theta + \beta\sin^2\theta \Longrightarrow (\alpha-\beta)\sin^2\theta = 0 \Longrightarrow \sin\theta = 0,\ \alpha \neq \beta$

$\therefore\ \theta = n\pi$

(ii) When $x = \beta$, $\beta = \alpha\cos^2\theta + \beta\sin^2\theta \Longrightarrow (\beta-\alpha)\cos^2\theta = 0 \Longrightarrow \cos\theta = 0,\ \alpha \neq \beta$

$\therefore\ \theta = n\pi + \frac{\pi}{2}$

Combining (i) and (ii), we take $\theta = 0$ and $\theta = \frac{\pi}{2}$ to be the new lower and upper limits.

$\therefore I = 2\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{(\beta-\alpha)^2 \sin^2\theta \cos^2\theta}} (\beta-\alpha)\cos\theta\sin\theta\, d\theta = 2\int_0^{\frac{\pi}{2}} d\theta = \underline{\underline{\boldsymbol{\pi}}}$, where $\alpha < \beta$.

When $\alpha > \beta$, $\sqrt{(\beta-\alpha)^2 \sin^2\theta \cos^2\theta} = -(\beta-\alpha)\cos\theta\sin\theta$

$\therefore I = \underline{\underline{\boldsymbol{-\pi}}}$

(b) When we use the substitution $t = \frac{1}{x}$,

$\int_a^b \frac{1}{t\sqrt{(t-a)(b-t)}} dt = \int_{1/a}^{1/b} \frac{1}{\frac{1}{x}\sqrt{\left(\frac{1}{x}-a\right)\left(b-\frac{1}{x}\right)}} \left(-\frac{dx}{x^2}\right) = -\int_{1/a}^{1/b} \frac{1}{\sqrt{(1-ax)(xb-1)}} dx$

$= -\int_{1/a}^{1/b} \frac{1}{\sqrt{ab\left(x-\frac{1}{a}\right)\left(\frac{1}{b}-x\right)}} dx = \underline{\underline{\frac{\boldsymbol{\pi}}{\sqrt{\mathbf{ab}}}}}$, from **(a)** and note that $\frac{1}{b} < \frac{1}{a}$.

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